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LETTER TO THE EDITOR

Complete positivity and dissipative factorized dynamics

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Abstract
We show that any Hermiticity and trace-preserving continuous semigroup $\{\gamma_t\}_{t\geqslant 0}$ in d dimensions is completely positive if and only if the semigroup $\{\gamma_t \otimes \gamma_t\}_{t\geqslant 0}$ is positivity-preserving.

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Complete positivity is a property of linear transformations of quantum states whose importance for physics is prominent in open quantum system dynamics and quantum communication.

The reduced dynamics of systems in (weak) interaction with their environment is usually generated by the equation of motion of Kossakowski–Lindblad form [1] and thus completely positive [2, 3]. However, it is still being debated whether such a constraint is physically necessary [4–8].

In contrast, in quantum communication theory only completely positive linear maps can describe local operations on quantum states [9]. Locality means that, given a bipartite system $A + B$ in a state ρ_{AB}, only the A-component is transformed according to $\gamma_A \otimes \mathbf{I}_B$, where $\mathbf{I}$ is the identity operation; then, if ρ_{AB} is entangled and γ_A not completely positive, $\gamma_A \otimes \mathbf{I}_B[\rho_{AB}]$ may develop negative eigenvalues and thus lose consistency as a physical state [10–12].

The same kind of argument is generally used to motivate why the reduced dynamics of an open quantum system A must be described by a semigroup of completely positive dynamical maps γ_t^A; if not, $\left(\gamma_t^A \otimes \mathbf{I}_B\right)[\rho_{AB}]$ may become physically inconsistent as the time evolution of an initial entangled state ρ_{AB} [1].

In this case, however, the partner system B is not, as in quantum communication theory, a concrete party, making up, for instance, a definite protocol for information transmission. Rather, B is a totally uncontrollable entity that may happen to have become entangled with the system of physical interest A; in this case one should not consider γ_t^A but $\gamma_t^A \otimes \mathbf{I}_B$ as the effective time evolution acting not on a state ρ_A, but on the effective initial state ρ_{AB} of the compound system $A + B$. It is the abstractness of such a setting that makes physically unpalatable the request of complete positivity in open quantum system dynamics [5].

More concretely, one may consider A and B as systems of the same kind in (weak) interaction with the same environment and thus evolving in time according to an approximate reduced dynamics of the form $\gamma_t \otimes \gamma_t$.

Actually, there exist some experimental setups where this is the case and, moreover, the compound system $A + B$ is initially prepared in a maximally entangled state ρ_{AB} [13, 14]. Then, the question is whether, for $(\gamma_t \otimes \gamma_t)[\rho_{AB}]$ to remain positive, γ_t need be completely positive or not.

In theorem 3 we shall prove that, in the case of A and B d-dimensional systems, this is indeed so: in order that $\gamma_t \otimes \gamma_t$ be positivity-preserving, γ_t must be completely positive. The argument in favour of the necessity of complete positivity for semigroup dynamics of open quantum systems is thus strengthened with respect to the argument based on $\gamma_t \otimes \mathbf{I}_B$.

Complete positivity is formulated as a property of linear maps Γ on algebras of operators X and by duality transferred to the corresponding transformations γ of quantum states, according to

$$\mathrm{Tr}(\rho\Gamma[X]) = \mathrm{Tr}(\gamma[\rho]X). \tag{1}$$

We shall consider states represented by density matrices ρ and restrict our attention to d-dimensional quantum systems so that the operators X will be represented by $d \times d$ matrices as well as the ρ.

Definition 1 [15]. *$\Gamma : M_d(\mathbf{C}) \to M_d(\mathbf{C})$ is completely positive if $\forall n \in \mathbf{N}$, the map $\Gamma \otimes \mathbf{I}_n$ preserves positivity on $M_d(\mathbf{C}) \otimes M_n(\mathbf{C})$, where $M_n(\mathbf{C})$ is any $n \times n$ matrix algebra and $\mathbf{I}_n$ the identity operation on it.*

In fact, one need not check all n but just $n = d$ as stated by a theorem of Choi [16].

Theorem 1. *$\Gamma : M_d(\mathbf{C}) \to M_d(\mathbf{C})$ is completely positive if and only if $\Gamma \otimes \mathbf{I}_d$ is positivity-preserving on $M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$.*

Remark 1. If the map $\Gamma : M_d(\mathbf{C}) \to M_d(\mathbf{C})$ is positivity-preserving, but not completely positive, then there is a positive $X \in M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$ such that $(\Gamma \otimes \mathbf{I}_d)[X]$ is not positive. If $|\psi\rangle$ is an eigenvector of $(\Gamma \otimes \mathbf{I}_d)[X]$ relative to a negative eigenvalue, via duality, we get

$$\mathrm{Tr}((\gamma \otimes \mathbf{I}_d)[|\psi\rangle\langle\psi|]X) = \langle\psi|(\Gamma \otimes \mathbf{I}_d)[X]|\psi\rangle < 0. \tag{2}$$

Therefore, the linear map $\gamma \otimes \mathbf{I}_d$, dual to $\Gamma \otimes \mathbf{I}_d$, does not preserve the positivity of $|\psi\rangle\langle\psi|$. Also, $|\psi\rangle$ must be entangled, for, if $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$, then $\gamma[|\psi_a\rangle\langle\psi_a|] \otimes |\psi_b\rangle\langle\psi_b|$ is positive.

As stated in the introduction, we are interested in semigroups of positive linear maps, $\{\gamma_t\}_{t\geqslant 0}$, on the states over $M_d(\mathbf{C})$. In particular, we shall be concerned with Hermiticity and trace-preserving, continuous semigroups on density matrices $\rho \in M_d(\mathbf{C})$,

$$\gamma_t \circ \gamma_s = \gamma_{t+s} = \gamma_s \circ \gamma_t \qquad \forall s, t \geqslant 0 \tag{3}$$

$$\mathrm{Tr}\,\gamma_t[\rho] = \mathrm{Tr}\,\rho \qquad \gamma_t[\rho]^\dagger = \gamma_t[\rho] \tag{4}$$

$$\lim_{t\to 0^+} \gamma_t[\rho] = \rho \tag{5}$$

the latter limit being understood in the trace-norm topology [2].

Proposition 1 [2]. *Any semigroup $\{\gamma_t\}$ satisfying (3)–(5) is generated by the equation:*

$$\partial_t\gamma_t[\rho] = -\mathrm{i}[H, \gamma_t[\rho]] + \sum_{a,b=1}^{d^2-1} c_{ab}\left[F_a\gamma_t[\rho]F_b^\dagger - \frac{1}{2}\left\{F_b^\dagger F_a, \gamma_t[\rho]\right\}\right] \tag{6}$$

where $H = H^\dagger$, $\mathrm{Tr}\, H = 0$; $\mathrm{Tr}\, F_a^\dagger F_b = \delta_{ab}$, $\mathrm{Tr}\, F_a = 0$, $F_{d^2} = \mathbf{1}_d/\sqrt{d}$ *and* $C = [c_{ab}]$ *is a* $(d^2-1)\times(d^2-1)$ *self-adjoint matrix depending solely on the choices of the traceless matrices* $\{F_a\}_{a=1}^{d^2-1}$.

If one asks the γ_t to be completely positive, that is dual to completely positive $\Gamma_t : M_d(\mathbf{C}) \to M_d(\mathbf{C})$, then

Theorem 2 [2]**.** *The semigroup* $\{\gamma_t\}_{t\geqslant 0}$ *generated by (6) consists of completely positive maps if and only if* $C = [c_{ab}]$ *is a positive-definite* $(d^2-1)\times(d^2-1)$ *matrix.*

Remark 2. If $C = [c_{ab}]$ is positive definite then it can be written $C = A^\dagger A$, $c_{ab} = \sum_{r=1}^{d^2-1} A^*_{ra} A_{rb}$, and

$$\sum_{a,b=1}^{d^2-1} c_{ab}\left[F_a\rho F_b^\dagger - \frac{1}{2}\left\{F_b^\dagger F_a, \rho\right\}\right] = \sum_{r=1}^{d^2-1}\left[V_r\rho V_r^\dagger - \frac{1}{2}\left\{V_r^\dagger V_r, \rho\right\}\right]$$

takes the Lindblad form [3] with $V_r = \sum_{a=1}^{d^2-1} A^*_{ra} F_a$. Conversely, given a generator in Lindblad form, developing $V_r = \sum_{a=1}^{d^2-1} v_{ra} F_a$ over a basis of traceless matrices F_a, one ends up with a generator as in (6) with $c_{ab} = \sum_{r=1}^{d^2-1} V_{ra} V^*_{rb}$ making for a $(d^2-1)\times(d^2-1)$ positive matrix $C = [c_{ab}]$.

Given a semigroup $\{\gamma_t\}_{t\geqslant 0}$ satisfying (3)–(5) and generated by (6), the justification why γ_t should be completely positive and thus the matrix $C = [c_{ab}]$ positive, is based on the fact that, otherwise, $\gamma_t \otimes \mathbf{I}_d$ would fail to preserve the positivity of entangled states on $M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$ (see remark 1).

However, while the first factor in $M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$ refers to a concrete open quantum system evolving in time according to (6), because of the interaction with a certain environment, the second factor represents a mere possibility of entanglement with anything described by a d-dimensional system and generically out of physical control.

Instead, we argue that complete positivity is necessary to avoid physical inconsistencies in compound systems consisting of two d-dimensional systems that interact with the same environment, but not among themselves, neither directly, nor indirectly, that is through the environment itself. In such a case, the two systems are expected to evolve according to the semigroups of linear maps $\gamma_t \otimes \gamma_t$, $t \geqslant 0$, where γ_t is the single open system dynamics obtained when only one of them is present in the environment.

A necessary reQuest for the physical consistency of such dynamics is that the $\gamma_t \otimes \gamma_t$ preserve the positivity of all separable and entangled states of the compound system, which now describe physically concrete and controllable settings.

Theorem 3. *If* $\{\gamma_t\}_{t\geqslant 0}$ *is a Hermiticity and trace-preserving continuous semigroup of linear maps over the states of* $M_d(\mathbf{C})$*, the semigroup* $\{\gamma_t \otimes \gamma_t\}_{t\geqslant 0}$ *of linear maps over the states of* $M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$ *is positivity-preserving if and only if* $\{\gamma_t\}_{t\geqslant 0}$ *is made of completely positive maps.*

The proof of theorem 3 will consist of several steps. We need just show the 'only if' part. Indeed, if γ_t is completely positive, $\gamma_t \otimes \mathbf{I}_d$ and $\mathbf{I}_d \otimes \gamma_t$ are both positive and such is the composite map $\gamma_t \otimes \gamma_t = (\gamma_t \otimes \mathbf{I}_d) \circ (\mathbf{I}_d \otimes \gamma_t)$: actually, it is completely positive (see e.g., proposition 4.23 in [15]).

Remark 3.

1. If the γ_t preserve the positivity of states of $M_d(\mathbf{C})$, $\gamma_t \otimes \gamma_t$ preserves the positivity of separable states of $M_d(\mathbf{C}) \otimes M_d(\mathbf{C})$: this follows by a straightforward adaptation of the argument in remark 1.

2. For generic positive linear maps γ on the states of $M_d(\mathbf{C})$, it does not follow that, if $\gamma \otimes \gamma$ is positivity-preserving, then γ is completely positive. A counter example is the transposition τ over $M_2(\mathbf{C})$: $\tau \otimes \tau$ is positivity-preserving, but τ is not completely positive. We note, however, that τ cannot be among the γ_t of a continuous semigroup over the states of $M_2(\mathbf{C})$ since it is not connected to the identity operation.
3. There are experimental situations that can be described by semigroups $\{\gamma_t \otimes \gamma_t\}_{t\geqslant 0}$. For instance, neutral mesons may be imagined to suffer from dissipative effects due to a noisy background determined by Planck's scale physics. As decay products of spin 1 resonances, these mesons are produced in maximally entangled states and, while independently flying apart back to back, they arguably evolve according to semigroups $\{\gamma_t \otimes \gamma_t\}_{t\geqslant 0}$ [12–14]. In such a context, whether $\gamma_t \otimes \gamma_t$ is positivity-preserving is crucial for concrete physical consistency.

Lemma 1. *If $\{\gamma_t\}_{t\geqslant 0}$ is a semigroup satisfying (3)–(5) and generated by (6), the semigroup $\{\gamma_t \otimes \gamma_t\}_{t\geqslant 0}$ consists of positivity-preserving maps only if*

$$\mathcal{L}_{\phi,\psi} \equiv \langle\phi|(L \otimes \mathbf{I}_d + \mathbf{I}_d \otimes L)[|\psi\rangle\langle\psi|]|\phi\rangle \geqslant 0 \tag{7}$$

for all orthogonal vector states $|\phi\rangle$, $|\psi\rangle$ in $\mathbf{C}^d$, where L is the generator on the right-hand side of (6) and $\mathbf{I}_d$ is the identity operation on $M_d(\mathbf{C})$.

Proof. From positivity preservation it follows that $\mathcal{G}_{\phi,\psi}(t) := \langle\phi|(\gamma_t \otimes \gamma_t)[|\psi\rangle\langle\psi|]|\phi\rangle \geqslant 0$, for all $|\phi\rangle$ and $|\psi\rangle \in \mathbf{C}^d \otimes \mathbf{C}^d$. Choosing $\langle\phi|\psi\rangle = 0$, if $\mathrm{d}\mathcal{G}_{\phi,\psi}(t)/\mathrm{d}t|_{t=0} < 0$, then $\mathcal{G}_{\phi,\psi}(t) \geqslant 0$ is violated in a neighbourhood of $t = 0$. Thus (7) follows. □

Lemma 2. *In the hypothesis of lemma 1, let $\{|j\rangle\}_{j=1}^d$ be an orthonormal basis of $\mathbf{C}^d$, and Φ, Ψ the $d \times d$ matrices $\Phi = [\varphi_{ij}]$, $\Psi = [\psi_{ij}]$ consisting of the coefficients of the expansion of $|\phi\rangle$ and $|\psi\rangle$ with respect to the basis $\{|j\rangle \otimes |k\rangle\}_{j,k=1}^d$ of $\mathbf{C}^d \otimes \mathbf{C}^d$. Then*

$$\mathcal{L}_{\phi,\psi} = \sum_{a,b=1}^{d^2-1} c_{ab}\big[\operatorname{Tr}(\Psi\Phi^\dagger F_a)\operatorname{Tr}\big(\Phi\Psi^\dagger F_b^\dagger\big) + \operatorname{Tr}((\Phi^\dagger\Psi)^T F_a)\operatorname{Tr}\big((\Psi^\dagger\Phi)^T F_b^\dagger\big)\big] \tag{8}$$

where $C = [c_{ab}]$ is the matrix of the coefficients and F_a, F_b the traceless matrices appearing in (6), while X^T denotes transposition of X with respect to the chosen basis.

Proof. Let $|\phi\rangle = \sum_{j,k=1}^d \varphi_{jk}|j\rangle \otimes |k\rangle$, $|\psi\rangle = \sum_{j,k=1}^d \psi_{jk}|j\rangle \otimes |k\rangle$; then, one calculates

$$\begin{aligned}\mathcal{L}_{\phi,\psi} &= \sum_{ij}\sum_{kl}\sum_{pr}\big(\varphi^*_{ij}\varphi_{kl}\psi_{pj}\psi^*_{rl} + \varphi^*_{ji}\varphi_{lk}\psi_{jp}\psi^*_{lr}\big)\langle i|L[|p\rangle\langle r|]|k\rangle \\ &= \sum_{ik}\sum_{pr}[(\Psi\Phi^\dagger)_{pi}(\Phi\Psi^\dagger)_{kr} + (\Phi^\dagger\Psi)_{ip}(\Psi^\dagger\Phi)_{rk}]\langle i|L[|p\rangle\langle r|]|k\rangle.\end{aligned} \tag{9}$$

The commutator and the anticommutator in the generator $L[\cdot]$ drop from equation (9); this is easily seen by noting that, given any $K \in M_d(\mathbf{C})$, $\langle i|(K|p\rangle\langle r|)|k\rangle = K_{ip}\delta_{rk}$. In (9), we can further sum over either $r = k$ or $i = p$; in either case, as $\langle\phi|\psi\rangle = 0$, we find $\operatorname{Tr}\Psi\Phi^\dagger = (\operatorname{Tr}\Phi\Psi^\dagger)^* = 0$ and the result follows. □

Lemma 3. *In the hypothesis of lemma 1, the matrix $C = [c_{ab}]$ in (8) must be positive definite.*

Proof. With any $\vec{w} = \{w_a\}_{a=1}^{d^2-1} \in \mathbf{C}^{d^2-1}$, we consider $W = \frac{1}{2}\sum_{a=1}^{d^2-1} w_a^* F_a$, which is a traceless $d \times d$ matrix. If matrices Ψ and Φ exist such that $\Phi\Psi^\dagger = W$ and $\Psi^\dagger\Phi = W^T$, then, from lemmas 1 and 2 and the orthogonality of the matrices F_a (compare theorem 2) it follows that

$$\mathcal{L}_{\phi,\psi} = \sum_{a,b=1}^{d^2-1} c_{ab} w_a^* w_b \geqslant 0 \tag{10}$$

hence the positivity of $C = [c_{ab}]$ and the proof of theorem 3. Any matrix W and its transpose with respect to the given basis, W^T, have the same elementary divisors; therefore, they are similar to the same canonical Jordan form and thus similar to each other [17]. Let Φ be such that $\Phi^{-1} W \Phi = W^T$, that is we take as vector $|\phi\rangle \in \mathbf{C}^d \otimes \mathbf{C}^d$ the one whose components ϕ_{ij} are the elements of the similarity matrix transforming the given W into its transpose W^T. It then follows that $\Psi^\dagger = \Phi^{-1} W$ and, moreover, $\Psi^\dagger \Phi = \Phi^{-1} W \Phi = W^T$, which is what we need. □

Remark 4.

1. In the proof of theorem 2 in [2], the maximally entangled state $|\phi_+\rangle = \frac{1}{d}\sum_{i=1}^d |i\rangle \otimes |i\rangle$ plays a crucial role; however, it concerns a generator of the form $L \otimes \mathbf{I}_d$ instead of $L \otimes \mathbf{I}_d + \mathbf{I}_d \otimes L$. In such a case, (8) reads

$$\mathcal{L}_{\phi,\psi} = \sum_{a,b=1}^{d^2-1} c_{ab}\, \mathrm{Tr}(\Psi\Phi^\dagger F_a)[\mathrm{Tr}(\Psi\Phi^\dagger F_b)]^*.$$

Choosing $\Phi = \Phi^\dagger = \mathbf{1}_d/d$ given by the components of $|\phi_+\rangle$ and $W = \Psi^\dagger = d\sum_{k=1}^{d^2-1} w_k^* F_k$, the result of theorem 2 in [2] immediately follows from our argument, for

$$\mathcal{L}_{\phi,\psi} = \sum_{a,b=1}^{d^2-1} c_{ab} w_a^* w_b \geqslant 0.$$

2. The choice of $\Phi = \mathbf{1}_d/d$ in the previous remark is fixed for all traceless matrices W and it is $\Psi^\dagger$ which is chosen to be W. This argument, however, does not work with generic $\mathcal{L}_{\phi,\psi}$ as in (7), for, in general, $W^T \neq W$. Nevertheless, when the F_a are self-adjoint and $C = [c_{ab}]$ symmetric, the choice $\Phi = \mathbf{1}_d/d$ suffices for proving theorem 3. In this case, positivity of $C = [c_{ab}]$ is checked against real vectors $\vec{w} \in \mathbf{R}^{d^2-1}$, so that one can restrict to self-adjoint $W = W^\dagger$ and choose a basis $\{|i\rangle\} \in \mathbf{C}^d$ such that W is diagonal; then $(\Psi^\dagger)^T = W^T = W$.
3. When $d = 2$, the maximally entangled Bell state $|\phi\rangle = (|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle)/\sqrt{2}$ plays for the generator $L \otimes \mathbf{I}_d + \mathbf{I}_d \otimes L$ the same role played by the symmetric state $|\phi_+\rangle$ for the generator $L \otimes \mathbf{I}_d$ in remark 4.1 [2]. Namely, given any traceless matrix $W = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$, we can choose $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Psi^\dagger = \Phi^{-1} W = \sqrt{2}\begin{pmatrix} -\gamma & \alpha \\ \alpha & \beta \end{pmatrix}$. It turns out that $\Psi^\dagger\Phi = \begin{pmatrix} -\alpha & -\gamma \\ -\beta & \alpha \end{pmatrix} = -W^T$ [18] and the minus sign is not felt by the expressions in (8).

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